Frequency Energy Multiplier Approach to Uniform Exponential Stability Analysis of Semi-discrete Scheme for a Schrödinger Equation under Boundary Feedback

Bao-Zhu Guo and Fu Zheng

***Abstract*—In this paper, we investigate the uniform exponential stability of a semi-discrete scheme for a Schrödinger equation under boundary feedback stabilizing control in the natural state space $L^2(0,1)$. This study is significant since a time domain energy multiplier that allows proving the exponential stability of this continuous Schrödinger system has not yet found, thus leading to a major mathematical challenge to semi-discretization of the PDE, an open problem for a long time. Although the powerful frequency domain energy multiplier approach has been used in proving exponential stability for PDEs since 1980s, its use to the *uniform* exponential stability of the semi-discrete scheme for PDEs has not been reported yet. The difficulty associated with the uniformity is that due to the parameter of the step size, it involves a family of operators in different state spaces that need to be considered simultaneously. Based on the Huang-Prüss frequency domain criterion for uniform exponential stability of a family of C_0-semigroups in Hilbert spaces, we solve this problem for the first time by proving the uniform boundedness for all the resolvents of these operators on the imaginary axis. The proof almost exactly follows the procedure for the exponential stability of the continuous counterpart, highlighting the advantage of this discretization method.**

***Index Terms*—Schrödinger equation, boundary damping, frequency domain multiplier, semi-discretization, uniform exponential stability.**

I. Introduction

Control systems described by partial differential equations (PDEs) is infinite-dimensional. Being such, its controller such as the observer-based feedback control is also infinite-dimensional. As a result, the discretization finds itself in almost all implementations of PDE control. Among many discretization methods is the finite-difference method which becomes popular due to its simplicity in principle and its appeal to engineers. One of the most commonly used discretization method is the so-called semi-discrete scheme which keeps time continuous while discretizing the spatial variable. It has been widely studied in literature. The main advantage of the semi-discrete scheme is that it results in an ordinary differential equation system, which control researchers are most familiar with. However, it has been acknowledged for a long time that the uniform exponential stability with respect to the spatial discrete step size cannot be guaranteed for classical semi-discrete schemes for PDEs, largely due to presence of high frequency spurious components. In addition, some other typical important control properties such as uniform observability and uniform exact controllability cannot be guaranteed either. The reason for this loss is that the spurious modes are only weakly damped in the process of semi-discretization. A detail account can be found in [24]. For wave equations, several remedies such as Tichonoff regularization [7], mixed-finite elements [2], [17], high frequency filtering [10], and non-uniform meshes [4], have been proposed to circumvent this difficulty. Among many these remedies, the numerical viscosity damping introduced in [19], [20] is the most popular. However, this approach brings a viscosity term artificially added into the classical discrete scheme. The coefficients of the numerical viscosity damping vary from PDE to PDE. Recently, a new natural semi-discrete scheme based on order reduction finite difference method was introduced in [13] and has been applied to different systems [8], [23]. This approach has the critical advantages that it guarantees the uniform exponential stability. In addition, as a natural semi-discrete scheme, it allows one to prove the uniform exponential stability in a manner parallel to its continuous PDE counterpart. Nevertheless, all the previous papers on this scheme involved construction of Lyapunov functional which the proof heavenly relies on, both for semi-discrete scheme and the continuous counterpart.

Construction of a suitable Lyapunov functional for a PDE relies on a time domain energy multiplier, which is not always available and its construction is most often very technical. In 1980s, a frequency domain energy multiplier approach was developed for the exponential stability initially for a single PDE ([15]). The approach is based on a frequency domain characterization for exponential stability of C_0-semigroup in Hilbert space. Originally developed independently in [9] and [18], the result of was proved later in [16] to be valid for uniform exponential stability of a family of C_0-semigroups in Hilbert spaces as well. Uniform admissibility and observability for the finite element space semi-discretizations of abstract Schrödinger system and second order infinite dimensional vibrating systems have also been developed [5], [6].

In this paper, we investigate the uniform exponential sta-

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bility of an order reduction semi-discrete scheme for a Schrödinger equation under boundary control by the frequency domain multiplier approach. It is significant because one cannot find a suitable time domain Lyapunov functional both for the continuous PDE and for its discrete scheme. This implies that successful approaches presented in [14], [13], [8], [23] cannot be applied here. As a matter of fact, in order to apply the Lyapunov method, the paper [14] has to consider the Schrödinger system in the high order state space $H^1(0,1)$, whereas our state space is the standard space $L^2(0,1)$. The problem in $L^2(0,1)$ has been open for quite a long time. Fairly speaking, this paper brings a new way to the proof of the uniform exponential stability of the semi-discrete scheme for PDEs. It is also worthy pointing out that the proofs for both continuous PDE and for the discrete counterpart are again analogous, demonstrating the advantage of the order reduction semi-discretization approach.

We proceed as follows. In the next section, Section II, we prove the exponential stability of the continuous PDE by the frequency domain multiplier method. Although it is the simplest PDE ever studied in the literature, it helps in constructing a frequency domain multiplier for its semi-discrete counterpart. In Section III, we design a semi-discretized scheme and obtain a family of finite-dimensional systems. In Section IV, the uniform exponential stability is developed by the frequency domain multiplier approach. We introduce the shadow element to help understand the numerical approximating scheme, which plays an important role in the proof of uniform stability. Some concluding remarks are included in Section V.

II. Stability of Schrödinger system via frequency domain multiplier

Consider the following Schrödinger equation under boundary control:

$$\begin{cases} w_t(x,t) = -iw_{xx}(x,t),\ t>0,\ x\in(0,1),\\ w(0,t)=0,\ t\geq 0,\\ w_x(1,t)=u(t),\ k>0,\ t\geq 0,\\ y(t)=w(1,t),\ t\geq 0,\\ w(x,0)=w^0(x),\ x\in[0,1], \end{cases} \tag{1}$$

where $u(\cdot)$ is the control, $y(\cdot)$ is the measured output and $w_0(\cdot)$ is the initial state. Under the proportional feedback control:

$$u(t) = -kiy(t),\ k>0, \tag{2}$$

the closed-loop system of (1) becomes

$$\begin{cases} w_t(x,t) = -iw_{xx}(x,t),\ t>0,\ x\in(0,1),\\ w(0,t)=0,\ t\geq 0,\\ w_x(1,t)=-kiw(1,t),\ k>0,\ t\geq 0,\\ w(x,0)=w^0(x),\ x\in[0,1], \end{cases} \tag{3}$$

We consider system (3) in the natural state space $L^2(0,1)$. Define the system operator of (3) as follows:

$$\begin{cases} Af=-if'',\forall f\in D(A),\\ D(A)=\{f\in L^2(0,1)|f\in H^2(0,1),\\ \qquad f(0)=0,\ f'(1)=-ikf(1)\}. \end{cases} \tag{4}$$

Then, (3) can be written as an evolution equation in $L^2(0,1)$:

$$\begin{cases} \dot{w}(\cdot,t)=Aw(\cdot,t),\\ w(x,0)=w_0(x). \end{cases} \tag{5}$$

It is seen that

$$\mathrm{Re}\langle Af,f\rangle_{L^2(0,1)} = \mathrm{Re}\int_0^1 -ikf''(x)\overline{f(x)}dx = -k|f(1)|^2, \tag{6}$$

which implies that A is dissipative. In addition, the operator A is invertible and

$$\begin{aligned} A^{-1}f(x) = {} & \frac{-kx\int_0^1 xf(x)dx}{1+ki} \\ & -i\int_x^1 (x-\tau)f(\tau)d\tau - i\int_0^1 xf(x)dx, \end{aligned} \tag{7}$$

which is bounded in $L^2(0,1)$. As a result, A generates a C_0-semigroup of contractions on $L^2(0,1)$ by the Lumer-Phillips theorem ([21, Theorem 3.8.4]) and since A^{-1} is compact, the spectrum of A consists of isolated eigenvalues only.

Furthermore, define the system energy for (3) as

$$E(t) = \frac{1}{2}\int_0^1 |w(x,t)|^2 dt, \tag{8}$$

which is non-increasing as a consequence of (6):

$$\dot{E}(t) = -k|w(1,t)|^2. \tag{9}$$

We point out that a different version of (3):

$$\begin{cases} w_t(x,t) = -iw_{xx}(x,t),\ t>0,\ x\in(0,1),\\ w(0,t)=0,\ t\geq 0,\\ w_x(1,t)=-kw_t(1,t),\ k>0,\ t\geq 0,\\ w(x,0)=w^0(x),\ x\in[0,1], \end{cases} \tag{10}$$

was investigated in [14], for which one can find a time domain energy multiplier, and a Lyapunov functional was then constructed to both system (10) and its semi-discrete counterpart. However, system (3) is a rather unusual system for which a time domain energy multiplier has been not found yet. A first exponential stability result of system (3) was proved by the Riesz basis approach in [12]. Although the Riesz basis is powerful and the result obtained is much deeper than the result obtained from the multiplier method; for instance the spectrum-determined growth condition is usually a consequence of the Riesz basis approach yet this is usually not the case with the multiplier method. Unfortunately, the Riesz basis is extremely difficult, at least at the moment, to be applied for the uniformly exponential stability of semi-discrete model for (3) developed in this paper.

In this paper, we use an alternative powerful method called the frequency energy multiplier method, which has been

developed for continuous PDEs over the last three decades ([15]). In stability analysis, we can almost give one-to-one correspondence from continuous system to its discrete counterpart using this method. Our approach is so powerful that can be applied to other PDEs as well. For notation simplicity, hereafter, we omit without confusion the obvious dependency in time and spatial domains. The $\mathbb{C}^n$ denotes the n-dimensional complex Euclidean space; the $\mathbb{N}^+$ stands for the set of the positive integer numbers; and $\mathbb{R}$ the set of real numbers.

Since A generates a C_0-semigroup of contractions on $L^2(0,1)$, a well-known result of Hung-Prüss theorem [9], [18] states that the C_0-semigroup generated by A is exponentially stable if and only if it possesses the following two properties:

1) Every imaginary number belongs to the resolvent set of A, that is, $i\mathbb{R}\subset\rho(A)$.
2) The inverse operator of $i\omega-A$ is uniformly bounded for all imaginary numbers, that is,

$$\sup_{\omega\in\mathbb{R}}\|(i\omega-A)^{-1}\|<\infty. \tag{11}$$

The property $i\mathbb{R}\subset\rho(A)$ is stated in the following Lemma 2.1.

Lemma 2.1: Let A be defined by (4). Then, $i\mathbb{R}\subset\rho(A)$.

Proof. If there exist $\beta\in\mathbb{R},\beta\neq 0$ and a nonzero $f\in D(A)$ such that $i\beta f=Af$, then

$$\begin{cases} i\beta f(x)=-if''(x),\\ f'(1)=-kif(1),\ f(0)=0.\end{cases} \tag{12}$$

Take the inner product with $f(\cdot)$ over $[0,1]$ on both sides of the first equation of (12) to obtain

$$i\beta\|f\|^2=-k|f(1)|^2+i\int_0^1|f'(x)|^2dx, \tag{13}$$

which gives $f(1)=0$ and hence $f'(1)=0$. This shows that (12) has only zero solution, a contradiction. ∎

Theorem 2.1: Let A be defined by (4). Then, (11) holds true. As a consequence, the C_0-semigroup e^{At} generated by A is exponentially stable in $L^2(0,1)$.

Proof. We prove by assuming contrary of (11) that there exit a sequence $\omega_n\to\infty$, $f_n\in D(A)$, $\|f_n\|=1$ that

$$\lim_{n\to\infty}\|(i\omega_n-A)f_n\|=0,$$

i.e.,

$$i\omega_n f_n+if_n''\to 0 \text{ in } L^2(0,1). \tag{14}$$

Since

$$\mathrm{Re}\langle(i\omega_n-A)f_n,f_n\rangle_{L^2(0,1)}=\mathrm{Re}\langle -Af_n,f_n\rangle=k|f_n(1)|^2\to 0, \tag{15}$$

by the boundary condition $f_n'(1)=-ikf_n(1)$, it gives

$$f_n'(1)\to 0. \tag{16}$$

From (14) and $\|f_n\|=1$, it follows that $\frac{f_n''(\cdot)}{\omega_n}$ is bounded in $L^2(0,1)$. By

$$|f_n'(x)-f_n'(1)|=\left|\int_1^x f_n''(s)ds\right|\leq\|f_n''\|,$$

it follows from (16) and $\omega_n\to\infty$ that

$$\frac{f_n'(\cdot)}{\omega_n} \text{ is bounded in } L^2(0,1). \tag{17}$$

Since

$$\mathrm{Re}\left\langle \omega_n f_n+f_n'',\frac{xf_n'}{\omega_n}\right\rangle_{L^2(0,1)}=\frac{|f_n(1)|^2}{2}-\frac{1}{2}\int_0^1|f_n(x)|^2dx$$
$$+\frac{1}{2\omega_n}|f_n'(1)|^2-\frac{1}{2\omega_n}\int_0^1|f_n'(x)|^2dx,$$

and

$$\left\langle \omega_n f_n+f_n'',\frac{xf_n'}{\omega_n}\right\rangle_{L^2(0,1)}\to 0,$$

we have by (15) and (16) that

$$\int_0^1|f_n(x)|^2dx+\frac{1}{\omega_n}\int_0^1|f_n'(x)|^2dx\to 0, \tag{18}$$

which shows that when $\omega_n>0$, $\|f_n\|^2\to 0$, which is a contradiction to $\|f_n\|=1$. On the other hand, since from (14) and $\omega_n\to\infty$, we have

$$\begin{aligned}
&\int_0^1\left|f_n(x)+\frac{f_n''(x)}{\omega_n}\right|^2dx\\
&=\int_0^1\left(f_n(x)+\frac{f_n''(x)}{\omega_n}\right)\left(\overline{f_n(x)}+\frac{\overline{f_n''(x)}}{\omega_n}\right)dx\\
&=\int_0^1\left[|f_n(x)|^2+\frac{|f_n''(x)|^2}{\omega_n^2}\right]dx\\
&\quad+\frac{1}{\omega_n}\int_0^1[f_n(x)\overline{f_n''(x)}+\overline{f_n(x)}f_n''(x)]dx\\
&=\int_0^1\left[|f_n(x)|^2+\frac{|f_n''(x)|^2}{\omega_n^2}\right]dx\\
&\quad+\frac{1}{\omega_n}[f_n(x)\overline{f_n'(x)}+\overline{f_n(x)}f_n'(x)]_0^1\\
&\quad-\frac{2}{\omega_n}\int_0^1|f_n'(x)|^2dx\to 0,
\end{aligned} \tag{19}$$

Substitute $f_n'(1)=-ikf_n(1)$ and $f_n(0)=0$ into (19), and use (15)-(16) to obtain

$$\int_0^1|f_n(x)|^2dx+\int_0^1\frac{|f_n''(x)|^2}{\omega_n^2}dx-\frac{2}{\omega_n}\int_0^1|f_n'(x)|^2dx\to 0. \tag{20}$$

which shows that when $\omega_n<0$, $\|f_n\|^2\to 0$, which is also a contradiction. ∎

III. Semi-discrete scheme of Schrödinger equation

In this section we apply the order reduction method to derive a semi-discrete scheme for (3). To this purpose, we introduce an intermediate variable $v(x,t)=w_x(x,t)$ to reduce the order of the spacial derivative of (3). In this way, the Schrödinger equation (3) can be rewritten as the following equivalent form:

$$\begin{cases} w_t(x,t)+iv_x(x,t)=0,\\ v(x,t)=w_x(x,t),\\ w(0,t)=0,\\ v(1,t)=-kiw(1,t),\\ w(x,0)=w^0(x).\end{cases} \tag{21}$$

The semi-discretization process is similar to [14]. For the sake of completeness, we sketch briefly the process. For fixed $N \in \mathbb{N}^+$, consider an equidistant partition of interval $[0,1]$:

$$0 = x_0 < x_1 < \cdots < x_j = jh < \cdots < x_{N+1} = 1,$$

where $h = \frac{1}{N+1}$ is the mesh size. Denote the sequence $\{u_j\}_0^{N+1}$ by $\{u_j\}_j$ and introduce respectively the average operator and the first-order finite difference operator as

$$u_{j+\frac{1}{2}} = \frac{u_j + u_{j+1}}{2}, \quad \delta_x u_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h}. \tag{22}$$

For the solutions $v(x,t)$ and $w(x,t)$ of (21), let $\{V_j(t)\}_j$ and $\{W_j(t)\}_j$ be grid functions at grids $\{x_j\}_j$, satisfying

$$V_j(t) = v(x_j,t), \quad W_j(t) = w(x_j,t), \quad 0 \le j \le N+1.$$

The first equation of system (21) holds at $(x_{j+\frac{1}{2}},t)$, i.e.,

$$w'(x_{j+\frac{1}{2}},t) + iv_x(x_{j+\frac{1}{2}},t) = 0,$$

where $x_{j+\frac{1}{2}} = (j+\frac{1}{2})h$. Hereafter the prime "$\prime$" represents the derivative with respect to time t. Replace the differential operator ∂_x with difference operator δ_x to get

$$W'_{j+\frac{1}{2}}(t) + i\delta_x V_{j+\frac{1}{2}}(t) = \mathscr{O}(h^2). \tag{23}$$

Similarly, for the second equation of system (21), it has

$$V_{j+\frac{1}{2}}(t) - \delta_x W_{j+\frac{1}{2}}(t) = \mathscr{O}(h^2). \tag{24}$$

By dropping the infinitesimal terms in (23) and (24), and replacing $W_j(t)$ and $V_j(t)$ by $w_j(t)$ and $v_j(t)$, respectively, we arrive at a semi-discretized finite difference scheme of system (21) as follows:

$$\begin{cases} w'_{j+\frac{1}{2}}(t) + i\delta_x v_{j+\frac{1}{2}}(t) = 0, \ 0 \le j \le N, \\ v_{j+\frac{1}{2}}(t) = \delta_x w_{j+\frac{1}{2}}(t), \ 0 \le j \le N, \\ v_{N+1}(t) = -kiw_{N+1}(t), \ t \ge 0 \\ w_0(t) = 0, \\ w_j(0) = w_j^0, \ 0 \le j \le N+1, \end{cases} \tag{25}$$

where $v_j(t)$ and $w_j(t)$ are grid functions at grids x_j $(0 \le j \le N+1)$, and w_j^0 is the approximation of the initial value $w^0(x_j)$.

Remark 3.1: The semi-discretized system (25) is a family of differentiation-algebra systems, which is called singular systems for which there are huge amount of references related to them. See for instance [3], [11], [22] and the references therein.

Now, we eliminate the intermediate variables $v_j(t)$ from (25). To this purpose, let

$$W_h(t) = (w_1(t), w_2(t), \cdots, w_{N+1}(t))^\top$$

be unknown variable of (25) and

$$V_h(t) = (v_0(t), v_1(t), \cdots, v_N(t))^\top$$

the auxiliary variable. We write (25) into vectorial form:

$$\begin{cases} D_h W'_h(t) = -iM_h V_h(t) - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ kh^{-1} w_{N+1}(t) \end{pmatrix}, \\ D_h^\top V_h(t) = -M_h^\top W_h(t) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ i2^{-1} k w_{N+1}(t) \end{pmatrix}, \\ W_h(0) = (w_0^0, w_1^0, \cdots, w_N^0)^\top, \end{cases} \tag{26}$$

where the matrices D_h and M_h are given by

$$D_h = \frac{1}{2} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & 1 & 1 & \\ & & & 1 & 1 \end{pmatrix}_{(N+1)\times(N+1)},$$
$$M_h = \frac{1}{h} \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}_{(N+1)\times(N+1)}. \tag{27}$$

Obviously, both D_h and M_h are invertible. The differential algebraic system (25) or (26) can be written as an evolution equation in $\mathbb{C}^{N+1}$:

$$\begin{cases} W'_h(t) = \mathscr{A}_h W_h(t), \quad W_h(t) \in \mathbb{Y}_h = \mathbb{C}^{N+1}, \\ W_h(0) = (w_1^0, w_2^0, \cdots, w_{N+1}^0)^\top \in \mathbb{Y}_h, \end{cases} \tag{28}$$

where $\mathscr{A}_h$ is defined by

$$\begin{aligned} &\mathscr{A}_h Y_h \\ &= D_h^{-1}\Big[iM_h \left(D_h^\top\right)^{-1} \left(M_h^\top Y_h - (0,\cdots,0,2^{-1}iky_{N+1})^\top\right) \\ &\quad - D_h^{-1}(0,\cdots,0,kh^{-1}y_{N+1})^\top \Big], \\ &\qquad \forall Y_h = (y_1, y_2, \cdots, y_{N+1})^\top \in \mathbb{C}^{N+1}. \end{aligned} \tag{29}$$

System (28) is naturally discussed in the state space $\mathbb{C}^{N+1}$. To relate $\mathbb{C}^{N+1}$ in (28) with the step size, we write $\mathbb{Y}_h = \mathbb{C}^{N+1}$ and define a new inner product for $\mathbb{Y}_h$:

$$\left\langle Y_h, \widetilde{Y}_h \right\rangle_{\mathbb{Y}_h} = h \left\langle D_h Y_h, D_h \widetilde{Y}_h \right\rangle, \forall Y_h, \widetilde{Y}_h \in \mathbb{Y}_h,$$

where $\langle \cdot,\cdot \rangle$ is the standard inner product of $\mathbb{C}^{N+1}$. For $Y_h = (y_1, y_2, \cdots, y_{N+1})^\top \in \mathbb{Y}_h$, we choose the vector

$$\begin{aligned} &Z_h = (z_0, z_1, \cdots, z_N)^\top \in \mathbb{C}^{N+1} \text{ satisfying} \\ &D_h^\top Z_h = -M_h^\top Y_h + (0,\cdots,0,2^{-1}iky_{N+1})^\top. \end{aligned} \tag{30}$$

We call Z_h the shadow element of Y_h, which can simplify significantly the notation in the later proofs.

The classical semi-discrete scheme is similar with (28) where the average operator $D_h = I_{N+1}$, i.e.,

$$\begin{cases} W'_h(t) = \hat{\mathscr{A}}_h W_h(t), \quad W_h(t) \in \mathbb{Y}_h = \mathbb{C}^{N+1}, \\ W_h(0) = (w_1^0, w_2^0, \cdots, w_{N+1}^0)^\top \in \mathbb{Y}_h, \end{cases} \tag{31}$$

in which the $\hat{\mathscr{A}}_h$ is defined by

$$\hat{\mathscr{A}}_h Y_h = iM_h\left(M_h^\top Y_h - (0,\cdots,0,2^{-1}iky_{N+1})^\top\right) - (0,\cdots,0,kh^{-1}y_{N+1})^\top. \tag{32}$$

At the end of this section we explain the significance of the discrete scheme (28). We plot two figures in Figures 1 and 2, respectively. Figure 1 depicts the maximal real parts of the eigenvalues of the classical semi-discrete scheme (31) with step size h, from which we see that the real parts of the eigenvalues approach zero. Figure 2 depicts the maximal real parts of the eigenvalues of the order reduction semi-discrete scheme (28) with the same step size, from which we see that the real parts of the eigenvalues approach a negative number. In both figures, we take $k=1$.

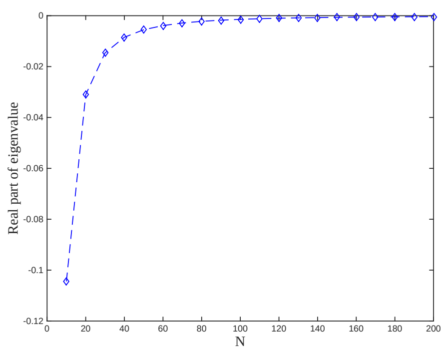

Fig. 1. Maximal real parts of eigenvalues of the semi-discrete scheme by classical method (31)

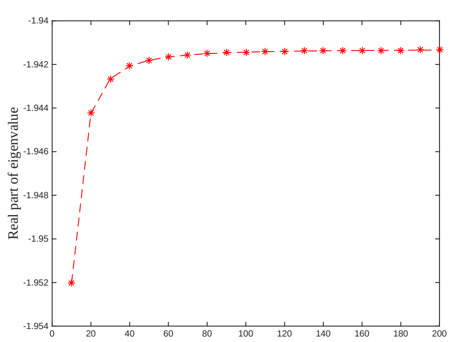

Fig. 2. Maximal real parts of eigenvalues of the semi-discrete scheme by order reduction method (28)

IV. Proof of Uniform Exponential Stability

This section is devoted to the proof of the uniform exponential stability of (28). To begin with, we first show that $\mathscr{A}_h$ is dissipative for every step size h.

Lemma 4.1: For the matrix $\mathscr{A}_h$ defined by (29), there holds

$$\mathrm{Re}\,\langle \mathscr{A}_h Y_h, Y_h\rangle_{\mathbb{Y}_h} = -k|y_{N+1}|^2,\ \forall\ Y_h \in \mathbb{Y}_h, \tag{33}$$

which implies that $\mathscr{A}_h$ is dissipative for every $h \in (0,1)$.

Proof. For $Y_h = (y_1, y_2, \cdots, y_{N+1}) \in \mathbb{Y}_h$, let $Z_h = (z_0, z_1, \cdots, z_N)$ be the shadow element of Y_h:

$$\begin{cases} D_h^\top Z_h = -M_h^\top Y_h + (0,\cdots,0,2^{-1}iky_{N+1})^\top, \\ \mathscr{A}_h Y_h = D_h^{-1}[-iM_h Z_h + (0,\cdots,0,kh^{-1}y_{N+1})^\top]. \end{cases} \tag{34}$$

Set $y_0 := 0$ and $z_{N+1} := -iky_{N+1}$ and introduce $\widetilde{Y}_h = (\widetilde{y}_1, \widetilde{y}_2, \cdots, \widetilde{y}_{N+1}) \in \mathbb{Y}_h$ such that $\mathscr{A}_h Y_h = \widetilde{Y}_h$. Then,

$$D_h^\top Z_h + (0,\cdots,0,2^{-1}z_{N+1})^\top = -M_h^\top Y_h, \tag{35}$$

which is equivalent to

$$z_{j+\frac{1}{2}} = \delta_x y_{j+\frac{1}{2}},\ \ j = 0,1,\cdots,N, \tag{36}$$

and

$$D_h \widetilde{Y}_h = -iM_h Z_h - (0,\cdots,0,ih^{-1}z_{N+1})^\top, \tag{37}$$

which is equivalent to

$$\widetilde{y}_{j+\frac{1}{2}} = -i\delta_x z_{j+\frac{1}{2}},\ \ j = 0,1,\cdots,N, \tag{38}$$

where in all (35) to (38), it was assumed that $\widetilde{y}_0 = 0$. Take the inner product between $\mathscr{A}_h Y_h$ and Y_h in $\mathbb{Y}_h$ by taking (36) and (38) into account to obtain

$$\begin{aligned}
&\mathrm{Re}\,\langle \mathscr{A}_h Y_h, Y_h\rangle_{\mathbb{Y}_h} = \mathrm{Re}\left\langle \widetilde{Y}_h, Y_h\right\rangle_{\mathbb{Y}_h} \\
&= \frac{h}{2}\left\langle D_h\widetilde{Y}_h, D_hY_h\right\rangle + \frac{h}{2}\left\langle D_hY_h, D_h\widetilde{Y}_h\right\rangle \\
&= \frac{h}{2}\sum_{j=0}^{N} \widetilde{y}_{j+\frac{1}{2}}\overline{y}_{j+\frac{1}{2}} + \frac{h}{2}\sum_{j=0}^{N} y_{j+\frac{1}{2}}\overline{\widetilde{y}}_{j+\frac{1}{2}},\ \text{(using (38))} \\
&= -\frac{hi}{2}\sum_{j=0}^{N}\delta_x z_{j+\frac{1}{2}}\overline{y}_{j+\frac{1}{2}} + \frac{hi}{2}\sum_{j=0}^{N} y_{j+\frac{1}{2}}\delta_x\overline{z}_{j+\frac{1}{2}} \\
&= -\frac{hi}{2}\sum_{j=0}^{N}\left[\delta_x z_{j+\frac{1}{2}}\overline{y}_{j+\frac{1}{2}}(t) + z_{j+\frac{1}{2}}\delta_x\overline{y}_{j+\frac{1}{2}}\right] \\
&\quad + \frac{hi}{2}\sum_{j=0}^{N}\left[y_{j+\frac{1}{2}}\delta_x\overline{z}_{j+\frac{1}{2}} + \delta_x y_{j+\frac{1}{2}}\overline{z}_{j+\frac{1}{2}}\right].\ \text{(using (36))}
\end{aligned} \tag{39}$$

A simple calculation shows that

$$\begin{aligned}
&-\frac{hi}{2}\sum_{j=0}^{N}\left[\delta_x z_{j+\frac{1}{2}}\overline{y}_{j+\frac{1}{2}} + z_{j+\frac{1}{2}}\delta_x\overline{y}_{j+\frac{1}{2}}\right] \\
&+\frac{hi}{2}\sum_{j=0}^{N}\left[y_{j+\frac{1}{2}}\delta_x\overline{z}_{j+\frac{1}{2}} + \delta_x y_{j+\frac{1}{2}}\overline{z}_{j+\frac{1}{2}}\right] \\
&= -\frac{i}{4}\sum_{j=0}^{N}\left[(z_{j+1}-z_j)(\overline{y}_{j+1}+\overline{y}_j) + (z_{j+1}+z_j)(\overline{y}_{j+1}-\overline{y}_j)\right] \\
&+\frac{i}{4}\sum_{j=0}^{N}\left[(y_{j+1}+y_j)(\overline{z}_{j+1}-\overline{z}_j) + (y_{j+1}-y_j)(\overline{z}_{j+1}(t)+\overline{z}_j)\right] \\
&= -\frac{i}{2}\sum_{j=0}^{N}[z_{j+1}\overline{y}_{j+1} - z_j\overline{y}_j] + \frac{i}{2}\sum_{j=0}^{N}[y_{j+1}\overline{z}_{j+1} - y_j\overline{z}_j] \\
&= \frac{i}{2}[z_0\overline{y}_0 - z_{N+1}\overline{y}_{N+1}] + \frac{i}{2}[y_{N+1}\overline{z}_{N+1} - y_0\overline{z}_0] \\
&= -k|y_{N+1}|^2.\ \ (\text{ using } -iky_{N+1} = z_{N+1} \text{ and } y_0 = 0)
\end{aligned} \tag{40}$$

The (39) and (40) leads to (33). ∎

Define the energy of (28) as

$$E_h(t)=\frac{h}{2}\sum_{j=0}^{N}\left|w_{j+\frac{1}{2}}(t)\right|^2=\frac{1}{2}\langle W_h(t),W_h(t)\rangle_{\mathbb{Y}_h}. \tag{41}$$

which is the discretization of the continuous energy (8). The following Lemma 4.2 is the discrete counterpart of (9), which is a consequence of (33).

Lemma 4.2: The $E_h(t)$ defined by (41) satisfies

$$\dot{E}_h(t)=-k|w_{N+1}(t)|^2. \tag{42}$$

The dissipativity of $\mathscr{A}_h$ implies that the spectral set $\sigma(\mathscr{A}_h)$ of $\mathscr{A}_h$ is contained in the closed left half-plane of the complex plane $\mathbb{C}$. Actually, we have more stronger result. Precisely, for any $0<h<1$, the spectral set $\sigma(\mathscr{A}_h)$ of $\mathscr{A}_h$ is contained in the open left half-plane of $\mathbb{C}$. This is the following Lemma 4.3.

Lemma 4.3: For every $h\in(0,1)$, $i\mathbb{R}\subset\rho(\mathscr{A}_h)$.

Proof. If there exist $\beta\in\mathbb{R}$ and nonzero $Y_h\in\mathbb{Y}_h$ such that $i\beta Y_h=\mathscr{A}_hY_h$, then it follows from (33) that

$$0=\mathrm{Re}\,\langle i\beta Y_h,\ Y_h\rangle_{\mathbb{Y}_h}=\mathrm{Re}\,\langle\mathscr{A}_hY_h,\ Y_h\rangle_{\mathbb{Y}_h}=-k|y_{N+1}|^2. \tag{43}$$

Replacing $\widetilde{Y}_h$ by $i\beta Y_h$ in (37), we obtain

$$\begin{cases}\beta y_{j+\frac{1}{2}}+\delta_x z_{j+\frac{1}{2}}=0, & 0\le j\le N,\\ z_{j+\frac{1}{2}}-\delta_x y_{j+\frac{1}{2}}=0, & 0\le j\le N,\end{cases} \tag{44}$$

where Z_h is the shadow element of Y_h defined in (34), $y_0=0$ and $z_{N+1}:=-iky_{N+1}$. Hence $z_{N+1}=y_{N+1}=0$ from (43). Setting $j=N$ in (44) yields

$$\beta hy_N=2z_N,\ z_N=-\frac{2}{h}y_N.$$

It follows that $y_N=z_N=0$ whenever βh^2+4 is nonzero. Under the condition $\beta h^2+4\neq 0$, suppose $z_{j+1}=y_{j+1}=0$ and solve (44) to arrive at $z_j=y_j=0$. This gives $Y_h=0$ by induction, which is a contradiction. On the other hand, whenever $\beta h^2+4=0$, it follows from (44) that

$$\begin{cases}\frac{1}{h}(y_{j+1}+y_j)=\frac{1}{2}(z_{j+1}-z_j),\ j=0,1,\cdots,N,\\ \frac{1}{h}(y_{j+1}-y_j)=\frac{1}{2}(z_{j+1}+z_j),\ j=0,1,\cdots,N,\end{cases} \tag{45}$$

which implies that

$$y_{j+1}=\frac{h}{2}z_{j+1},\ y_j=-\frac{h}{2}z_j,\ j=0,1,\cdots,N. \tag{46}$$

This, combining with $y_0=0$ and $y_{N+1}=0$, gives $y_j=0$ ($j=1,2,\cdots,N$) which is also a contradiction. This completes the proof of the lemma. ∎

The following lemma comes from [13].

Lemma 4.4: Let $\{u_i\}_i$, $\{v_i\}_i$ and $\{w_i\}_i$ be the sequences of complex numbers. Then,

$$\begin{aligned}&\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}-u_i)(v_{i+1}+v_i)(w_{i+1}+w_i)\\ &+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}-u_i)(v_{i+1}-v_i)(w_{i+1}-w_i)\\ &+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}+u_i)(v_{i+1}-v_i)(w_{i+1}+w_i)\\ &+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}+u_i)(v_{i+1}+v_i)(w_{i+1}-w_i)\\ &=u_{N+1}v_{N+1}w_{N+1}-u_0v_0w_0.\end{aligned} \tag{47}$$

The following uniformly stability criterion which was presented in [15] or [1] will be used in the proof of our main result Theorem 4.2 later.

Theorem 4.1: Let $h^*>0$ and let $\{S_h(t)\}_{h\in(0,h^*)}$ be a family of semigroups of contractions on the Hilbert space H_h, and let $\widetilde{A}_h$ be the corresponding infinitesimal generators. The family $\{S_h(t)\}$ is uniformly exponentially stable if and only if the following two conditions are fulfilled:

- For every $h\in(0,h^*)$, $i\mathbb{R}\subset\rho(\widetilde{A}_h)$;
- $\sup_{h\in(0,h^*),\beta\in\mathbb{R}}\|(i\beta I-\widetilde{A}_h)^{-1}\|<\infty$.

Now, we are in a position to give the main result of this paper.

Theorem 4.2: For the matrices $\mathscr{A}_h$ defined by (29), the corresponding family of C_0-semigroups $T_h(t)$ generated by $\mathscr{A}_h$ is uniformly exponentially stable, that is, there exist two constants $M>0$ and $\omega>0$ independent of $h\in(0,1)$ such that

$$\|T_h(t)\|\le Me^{-\omega t},\ \forall t\ge 0. \tag{48}$$

Proof. The proof is based on Theorem 4.1. Notice that by Lemma 4.1, for every $h\in(0,1)$, $T_h(t)$ is a C_0-semigroup of contractions. The fact that $\mathscr{A}_h$ satisfies the first condition of Theorem 4.1 has been claimed by Lemma 4.3. In order to show that the family $\mathscr{A}_h$ satisfies the second condition of Theorem 4.1, we prove by contradiction. If the second condition of Theorem 4.1 is false, then there exist a sequence $\beta_n\in\mathbb{R}$, $h_n\in(0,1)$, and $Y_{h_n}^n\in\mathbb{Y}_{h_n}, \|Y_{h_n}^n\|_{\mathbb{Y}_{h_n}}=1$ such that

$$\|U_{h_n}^n\|_{\mathbb{Y}_{h_n}}\le n^{-1},\ U_{h_n}^n=(i\beta_nI_{h_n}-\mathscr{A}_{h_n})Y_{h_n}^n. \tag{49}$$

By the Cauchy-Schwartz inequality, it follows from (49) and (33) that

$$\begin{aligned}\mathrm{Re}\,\langle U_{h_n}^n,Y_{h_n}^n\rangle_{\mathbb{Y}_{h_n}}&=-\mathrm{Re}\,\langle\mathscr{A}_{h_n}Y_{h_n}^n,Y_{h_n}^n\rangle_{\mathbb{Y}_{h_n}}\\ &=k|y_{N_n+1}^n|^2\le n^{-1}.\end{aligned} \tag{50}$$

Let

$$Z_{h_n}^n=(z_0^n,z_1^n,\cdots,z_{N_n}^n)^\top\in\mathbb{Z}_{h_n}$$

be the shadow element of $Y_{h_n}^n=(y_1^n,y_2^n,\cdots,y_{N_n+1}^n)^\top$(see(30)), $U_{h_n}^n=(u_1^n,u_2^n,\cdots,u_{N_n+1}^n)^\top$ with $h_n(N_n+1)=1$. Set artificially $u_0^n=y_0^n=0$ and $z_{N_n+1}^n=-iky_{N_n+1}^n$ to unify the notation of

$u^n_{j+\frac{1}{2}}$ and $\delta_x z^n_{j+\frac{1}{2}}$ for $j=0,1,\cdots,N_n$. Then, it follows from (49) that

$$\begin{cases} D_{h_n}U^n_{h_n} = i\beta_n D_{h_n}Y^n_{h_n} + iM_{h_n}Z^n_{h_n} + (0,\cdots,0,ih^{-1}z^n_{N_n+1})^\top, \\ -M^\top_{h_n}Y^n_{h_n} = D^\top_{h_n}Z^n_{h_n} + (0,\cdots,0,2^{-1}z^n_{N_n+1})^\top, \end{cases} \tag{51}$$

or in vector form:

$$\begin{cases} \begin{pmatrix} u^n_{0+\frac{1}{2}} \\ u^n_{1+\frac{1}{2}} \\ \vdots \\ u^n_{N_n+\frac{1}{2}} \end{pmatrix} = i\beta_n \begin{pmatrix} y^n_{0+\frac{1}{2}} \\ y^n_{1+\frac{1}{2}} \\ \vdots \\ y^n_{N_n+\frac{1}{2}} \end{pmatrix} + i \begin{pmatrix} \delta_x z^n_{0+\frac{1}{2}} \\ \delta_x z^n_{1+\frac{1}{2}} \\ \vdots \\ \delta_x z^n_{N_n+\frac{1}{2}} \end{pmatrix}, \\ \begin{pmatrix} z^n_{0+\frac{1}{2}} \\ z^n_{1+\frac{1}{2}} \\ \vdots \\ z^n_{N_n+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \delta_x y^n_{0+\frac{1}{2}} \\ \delta_x y^n_{1+\frac{1}{2}} \\ \vdots \\ \delta_x y^n_{N_n+\frac{1}{2}} \end{pmatrix}. \end{cases} \tag{52}$$

The proof will be split into three claims and each claim corresponds to that in the proof of stability of PDE. Clam 1 corresponds to $\omega_n \to \infty$ in the proof of Theorem 2.1.

Cliam 1: $|\beta_n| \ge C' > 0$ for some constant C' independent of $n \in \mathbb{N}^+$.

Suppose by contrary that the sequence $\{\beta_n\}$ contains a subsequence which is still denoted by $\{\beta_n\}$ itself without loss of generality converging to zero. Since $\|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}} = 1$ and $\|U^n_{h_n}\|_{\mathbb{Y}_{h_n}} \le n^{-1}$, it follows from (52) that

$$\begin{aligned} & h_n \sum_{j=0}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2 = h_n \sum_{j=0}^{N_n} \left|u^n_{j+\frac{1}{2}} - i\beta_n y^n_{j+\frac{1}{2}}\right|^2 \\ & \le 2h_n \sum_{j=0}^{N_n} \left|u^n_{j+\frac{1}{2}}\right|^2 + 2\beta_n^2 h_n \sum_{j=0}^{N_n} \left|y^n_{j+\frac{1}{2}}\right|^2 \\ & = 2\|U^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} + 2\beta_n^2 \|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} \le 2n^{-2}, \end{aligned} \tag{53}$$

which holds for all sufficiently large n. On the other hand, by some simple operations, we get

$$\begin{aligned} & |z^n_j - z^n_{N_n+1}|^2 = |z^n_j - z^n_{j+1} + z^n_{j+1} - z^n_{j+2} + z^n_{j+2} \cdots - z^n_{N_n+1}|^2 \\ & = \left|\sum_{l=j}^{N_n} (z^n_{l+1} - z^n_l)\right|^2 \le \left(\sum_{l=j}^{N_n} |1|^2\right)\left(\sum_{l=j}^{N_n} |z^n_{l+1} - z^n_l|^2\right) \\ & \le (N_n+1)\left(\sum_{l=0}^{N_n} |z^n_{l+1} - z^n_l|^2\right) \\ & = h_n \sum_{j=1}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2, \quad j=0,1,\cdots,N_n, \end{aligned} \tag{54}$$

in which $h_n(N_n+1)=1$ is used in the last step, and for $j=0,1,\cdots,N_n$

$$|z^n_j| \le |z^n_j - z^n_{N_n+1}| + |z^n_{N_n+1}| \le \sqrt{h_n \sum_{j=1}^{N_n} |\delta_x z^n_{j+\frac{1}{2}}|^2} + |z^n_{N_n+1}|.$$

This inequality, together with $z^n_{N_n+1} = -iky^n_{N_n+1}$ and (50)-(53), implies that for each $j=0,1,\cdots,N_n$, $|z^n_j|^2 = \mathscr{O}(n^{-1})$. Therefore, in light of $h_n(N_n+1)=1$ and the second identity of (52),

$$\begin{aligned} & h_n \sum_{j=0}^{N_n} \left|z^n_{j+\frac{1}{2}}\right|^2 \le \frac{h_n}{2} \sum_{j=0}^{N_n} \left(\left|z^n_{j+1}\right|^2 + \left|z^n_j\right|^2\right) \\ & \le h_n(N_n+1)\left|\sqrt{h_n \sum_{j=1}^{N_n} |\delta_x z^n_{j+\frac{1}{2}}|^2} + |z^n_{N_n+1}|\right|^2 \\ & \le h_n(N_n+1)Cn^{-1} = \mathscr{O}(n^{-1}). \end{aligned} \tag{55}$$

Thus, the deducing process from (53) to (55) tells us that

$$h_n \sum_{j=0}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2 = \mathscr{O}(n^{-2}),$$

which implies that

$$h_n \sum_{j=0}^{N_n} \left|z^n_{j+\frac{1}{2}}\right|^2 = \mathscr{O}(n^{-1}).$$

By noticing the second identity of (52), we have

$$h_n \sum_{j=0}^{N_n} \left|\delta_x y^n_{j+\frac{1}{2}}\right|^2 = h_n \sum_{j=0}^{N_n} \left|z^n_{j+\frac{1}{2}}\right|^2,$$

which means, by (55) that

$$h_n \sum_{j=0}^{N_n} \left|\delta_x y^n_{j+\frac{1}{2}}\right|^2 = \mathscr{O}(n^{-1}).$$

Similarly, repeating the procedure from (53) to (55), for $Y^n_{h_n}$, we obtain

$$\|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} = h_n \sum_{j=0}^{N_n} \left|y^n_{j+\frac{1}{2}}\right|^2 = \mathscr{O}(n^{-1/2}),$$

which leads to a contradiction. Thus, the sequence $\{\beta_n\}$ cannot contain a subsequence converging to zero. We can therefore assume that $|\beta_n| \ge C' > 0$ for some constant C' independent of $n \in \mathbb{N}^+$.

The second claim is the discrete counterpart of (18) but with two extra terms

$$\frac{h_n^3}{4\beta_n} \sum_{j=0}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2 + \frac{h_n^3}{4} \sum_{j=0}^{N_n} \left|\delta_x y^n_{j+\frac{1}{2}}\right|^2$$

which play important roles in our proofs.

Claim 2: the following (56) holds true:

$$\begin{aligned} & \|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} + \frac{1}{\beta_n} h_n \|\Sigma_{h_n} \widehat{Z}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}} \\ & + \frac{h_n^2}{4\beta_n} h_n \|\Delta_{h_n} \widehat{Z}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}} + \frac{h_n^2}{4} h_n \|\Delta_{h_n} \widehat{Y}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}} \\ & = h_n \sum_{j=0}^{N_n} \left|y^n_{j+\frac{1}{2}}\right|^2 + \frac{h_n}{\beta_n} \sum_{j=0}^{N_n} \left|z^n_{j+\frac{1}{2}}\right|^2 \\ & + \frac{h_n^3}{4\beta_n} \sum_{j=0}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2 + \frac{h_n^3}{4} \sum_{j=0}^{N_n} \left|\delta_x y^n_{j+\frac{1}{2}}\right|^2 \\ & = \mathscr{O}(n^{-1}), \end{aligned} \tag{56}$$

where $\|\cdot\|_{\mathbb{C}^{N_n+2}}$ denotes the standard norm of $\mathbb{C}^{N_n+2}$ and

$$\begin{aligned}
&\widehat{Z}^n_{h_n}=(z_0,z_1,\cdots,z_{N_n+1})^\top=\left((Z^n_{h_n})^\top,z_{N_n+1}\right)^\top,\\
&\widehat{Y}^n_{h_n}=(0,y_1,\cdots,y_{N_n+1})^\top=\left(0,(Y^n_{h_n})^\top\right)^\top,\\
&\Sigma_h=\frac{1}{2}\begin{pmatrix}1&1&&&\\&\ddots&\ddots&&\\&&1&1&\\&&&1&1\end{pmatrix}_{(N+2)\times(N+1)},\\
&\Delta_h=\frac{1}{h}\begin{pmatrix}-1&1&&&\\&-1&1&&\\&&\ddots&\ddots&\\&&&-1&1\end{pmatrix}_{(N+2)\times(N+1)}.
\end{aligned}\tag{57}$$

Actually, it follows from (49), (52), and $\|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}}=1$ that $\beta_n^{-2}h_n\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac{1}{2}}\right|^2$ is uniformly bounded with respect to $n\in\mathbb{N}^+$ because

$$\frac{1}{\beta_n}\begin{pmatrix}\delta_x z^n_{0+\frac{1}{2}}\\ \delta_x z^n_{1+\frac{1}{2}}\\ \vdots\\ \delta_x z^n_{N_n+\frac{1}{2}}\end{pmatrix}=-\begin{pmatrix}y^n_{0+\frac{1}{2}}\\ y^n_{1+\frac{1}{2}}\\ \vdots\\ y^n_{N_n+\frac{1}{2}}\end{pmatrix}-\frac{i}{\beta_n}\begin{pmatrix}u^n_{0+\frac{1}{2}}\\ u^n_{1+\frac{1}{2}}\\ \vdots\\ u^n_{N_n+\frac{1}{2}}\end{pmatrix}.$$

By (55) and **Claim 1**,

$$\beta_n^{-2}h_n\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2$$

is also uniformly bounded with respect to $n\in\mathbb{N}^+$. Let $x^n_j=jh_n$ for $j=0,1,\cdots,N_n+1$ and consider the following estimates:

$$\begin{aligned}
&\left|h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}(\beta_n y^n_{j+\frac{1}{2}}+\delta_x z^n_{j+\frac{1}{2}})\frac{\overline{z}^n_{j+\frac{1}{2}}}{\beta_n}\right|^2\\
=&\left|h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}u^n_{j+\frac{1}{2}}\frac{\overline{z}^n_{j+\frac{1}{2}}}{\beta_n}\right|^2\\
\le&\left(\sum_{j=0}^{N_n}\left|\sqrt{h_n}u^n_{j+\frac{1}{2}}\right|\left|\sqrt{h_n}\frac{z^n_{j+\frac{1}{2}}}{\beta_n}\right|\right)^2\\
\le&\left(h_n\sum_{j=0}^{N_n}\left|u^n_{j+\frac{1}{2}}\right|^2\right)\left(\beta_n^{-2}h_n\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2\right)\\
=&\|U^n_{h_n}\|^2_{\mathbb{Y}_{h_n}}\left(\beta_n^{-2}h_n\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2\right)\\
=&\mathscr{O}(n^{-2}),
\end{aligned}\tag{58}$$

where (49) and (52) were used. On the other hand, by the second identity of (52), we have

$$\begin{aligned}
&h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}(\beta_n y^n_{j+\frac{1}{2}}+\delta_x z^n_{j+\frac{1}{2}})\frac{\overline{z}^n_{j+\frac{1}{2}}}{\beta_n}\\
=&h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}y^n_{j+\frac{1}{2}}\delta_x\overline{y}^n_{j+\frac{1}{2}}\\
&+\beta_n^{-1}h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}\delta_x z^n_{j+\frac{1}{2}}\overline{z}^n_{j+\frac{1}{2}}.
\end{aligned}\tag{59}$$

Applying Lemma 4.4 to the two terms of the right hand side of (59) and noticing $x^n_{N_n+1}=1$, $x^n_0=0$, $x^n_{j+1}-x^n_j=h_n$, it is easy to obtain

$$\begin{aligned}
&2\mathrm{Re}\left(h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}y^n_{j+\frac{1}{2}}\delta_x\overline{y}^n_{j+\frac{1}{2}}\right)\\
=&h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}y^n_{j+\frac{1}{2}}\delta_x\overline{y}^n_{j+\frac{1}{2}}+h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}\overline{y}^n_{j+\frac{1}{2}}\delta_x y^n_{j+\frac{1}{2}}\\
=&\frac{1}{4}\sum_{j=0}^{N_n}(x^n_{j+1}+x^n_j)(y^n_{j+1}+y^n_j)(\overline{y}^n_{j+1}-\overline{y}^n_j)\\
&+\frac{1}{4}\sum_{j=0}^{N_n}(x^n_{j+1}+x^n_j)(y^n_{j+1}-y^n_j)(\overline{y}^n_{j+1}+\overline{y}^n_j)\\
=&|y_{N_n+1}|^2-h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac{1}{2}}\right|^2-\frac{h_n^3}{4}\sum_{j=0}^{N_n}\left|\delta_x y^n_{j+\frac{1}{2}}\right|^2,
\end{aligned}\tag{60}$$

and

$$\begin{aligned}
&2\mathrm{Re}\left(h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}z^n_{j+\frac{1}{2}}\delta_x\overline{z}^n_{j+\frac{1}{2}}\right)\\
=&h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}z^n_{j+\frac{1}{2}}\delta_x\overline{z}^n_{j+\frac{1}{2}}+h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}\overline{z}^n_{j+\frac{1}{2}}\delta_x z^n_{j+\frac{1}{2}}\\
=&\frac{1}{4}\sum_{j=0}^{N_n}(x^n_{j+1}+x^n_j)(z^n_{j+1}+z^n_j)(\overline{z}^n_{j+1}-\overline{z}^n_j)\\
&+\frac{1}{4}\sum_{j=0}^{N_n}(x^n_{j+1}+x^n_j)(\overline{z}^n_{j+1}+\overline{z}^n_j)(z^n_{j+1}-z^n_j)\\
=&|z_{N_n+1}|^2-h_n\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2-\frac{h_n^3}{4}\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac{1}{2}}\right|^2.
\end{aligned}\tag{61}$$

By (59)-(61), it follows that

$$\begin{aligned}
&h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac{1}{2}}\right|^2+\frac{h_n^3}{4}\sum_{j=0}^{N_n}\left|\delta_x y^n_{j+\frac{1}{2}}\right|^2\\
&+\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2+\frac{h_n^3}{4\beta_n}\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac{1}{2}}\right|^2\\
=&-2\mathrm{Re}\left(h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}(\beta_n y^n_{j+\frac{1}{2}}+\delta_x z^n_{j+\frac{1}{2}})\frac{\overline{z}^n_{j+\frac{1}{2}}}{\beta_n}\right)\\
&+|y_{N_n+1}|^2+\beta_n^{-1}|z_{N_n+1}|^2,
\end{aligned}\tag{62}$$

which proves (56) by (50), (58) and $z_{N_n+1}=-iky_{N_n+1}$.

The third claim is perfectly the discrete counterpart of (20).

Claim 3: The following (63) holds true:

$$\begin{aligned}&\|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}}+\frac{1}{\beta_n^2}h_n\|\Delta_{h_n}\widehat{Z}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}}-\frac{2}{\beta_n}h_n\|\Sigma_{h_n}\widehat{Z}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}}\\&=h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac12}\right|^2+\frac{h_n}{\beta_n^2}\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac12}\right|^2-\frac{2h_n}{\beta_n}\sum_{j=0}^{N_n}\left|z^n_{j+\frac12}\right|^2\\&=\mathscr{O}(n^{-2}),\end{aligned}\tag{63}$$

in which Σ_{h_n} and Δ_{h_n} are defined in (57) and $\|\cdot\|_{\mathbb{C}^{N_n+2}}$ denotes the standard norm of $\mathbb{C}^{N_n+2}$.

Actually, from (52), we have

$$\begin{aligned}&\frac{\|U^n_{h_n}\|^2_{\mathbb{Y}_{h_n}}}{\beta_n^2}\\&=\frac{h_n}{\beta_n^2}\sum_{j=0}^{N_n}\left|u^n_{j+\frac12}\right|^2=h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac12}+\frac{\delta_x z^n_{j+\frac12}}{\beta_n}\right|^2\\&=h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac12}\right|^2+\frac{h_n}{\beta_n^2}\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac12}\right|^2\\&\quad+\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+\frac12}\delta_x z^n_{j+\frac12}+y^n_{j+\frac12}\delta_x\overline{z}^n_{j+\frac12}).\end{aligned}\tag{64}$$

On the other hand, it follows from the second identity of (52) that $z^n_{j+\frac12}=\delta_x y^n_{j+\frac12}$ and

$$\begin{aligned}&\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+\frac12}\delta_x z^n_{j+\frac12}+y^n_{j+\frac12}\delta_x\overline{z}^n_{j+\frac12})+\frac{2h_n}{\beta_n}\sum_{j=0}^{N_n}\left|z^n_{j+\frac12}\right|^2\\&=\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+\frac12}\delta_x z^n_{j+\frac12}+\delta_x\overline{y}^n_{j+\frac12}z^n_{j+\frac12})\\&\quad+\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\delta_x y^n_{j+\frac12}\overline{z}^n_{j+\frac12}+y^n_{j+\frac12}\delta_x\overline{z}^n_{j+\frac12})\\&=\frac{1}{2\beta_n}\sum_{j=0}^{N_n}[(\overline{y}^n_{j+1}+\overline{y}^n_j)(z^n_{j+1}-z^n_j)+(\overline{y}^n_{j+1}-\overline{y}^n_j)(z^n_{j+1}+z^n_j)]\\&\quad+\frac{1}{2\beta_n}\sum_{j=0}^{N_n}[(y^n_{j+1}+y^n_j)(\overline{z}^n_{j+1}-\overline{z}^n_j)+(y^n_{j+1}-y^n_j)(\overline{z}^n_{j+1}+\overline{z}^n_j)]\\&=\frac{1}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+1}z^n_{j+1}-\overline{y}^n_jz^n_j)+\frac{1}{\beta_n}\sum_{j=0}^{N_n}(y^n_{j+1}\overline{z}^n_{j+1}-y^n_j\overline{z}^n_j)\\&=\frac{1}{\beta_n}[\overline{y}^n_{N_n+1}z^n_{N_n+1}+y^n_{N_n+1}\overline{z}^n_{N_n+1}-\overline{y}^n_0z^n_0-y^n_0\overline{z}^n_0]=0,\end{aligned}$$

where $z^n_{N_n+1}=-iky^n_{N_n+1}$ and $y^n_0=0$ were used in the last step. Hence

$$\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+\frac12}\delta_x z^n_{j+\frac12}+y^n_{j+\frac12}\delta_x\overline{z}^n_{j+\frac12})=-\frac{2h_n}{\beta_n}\sum_{j=0}^{N_n}\left|z^n_{j+\frac12}\right|^2.\tag{65}$$

Plugging (65) into (64) and using $\|U^n_{h_n}\|_{\mathbb{Y}_{h_n}}\le n^{-1}$ and **Claim 1**, we arrive at (63).

Finally, if $\beta_n>0$, we have $\|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}}=\mathscr{O}(n^{-1})$ from (56), which is a contradiction to $\|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}}=1$. When $\beta_n<0$, $\|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}}=\mathscr{O}(n^{-1})$ by virtue of (63), which is also a contradiction. We therefore complete the proof of the theorem. ∎

V. Concluding Remarks

In this paper, the uniform approximation of exponential stability of a one-dimensional Schrödinger equation is investigated. We introduce an order reduction space semi-discretized finite difference scheme for approximating uniformly the exponentially stable closed-loop system. Although the scheme has been applied to certain PDEs in previous works, they all share a common feature that it is possible to find a suitable Lyapunov functional for the closed-loop systems, both for the continuous system and its discrete counterpart. However, for the system considered in this paper, it is a longstanding problem that in the natural state space $L^2(0,1)$, even for the continuous system, the time domain energy multiplier has not been found. This makes the convergence of the semi-discrete scheme of this PDE be open for a long time. This paper is the first work that applies the frequency domain multiplier approach to the uniformly exponential convergence of semi-discretized PDE system. The convergence of the discrete scheme to continuous system is not included because it is a standard procedure and can be followed analogously from [14] and many other papers. Considering it is difficult to find a time domain energy multiplier for many other PDEs, the approach presented in this paper has significant potentials in applying to other PDEs.

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